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Characterizations of Transversal Matroids and Their Presentations

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An algorithm is presented for determining whether or not a matroid is a transversal matroid. If the matroid is a transversal matroid, the algorithm furnishes an explicit determination of the maximal presentation (which therefore must be unique). From this we obtain necessary and sufficient conditions for a matroid to be a transversal matroid and two characterizations of the presentations of a given transversal matroid. We also evaluate the cardinalities of the members of presentations of a transversal matroid in terms of the cardinalities of the members of the maximal presentation and the ranks of the complements of each. Numerous other consequences are derived.

1. INTRODUCTION

Transversal matroids are those matroids where the independent sets can be considered as the partial transversals of a family of sets. For a given transversal matroid such a family of sets is called a presentation of the matroid. Transversal matroids were discovered several years ago by Edmonds and Fulkerson [7], and since then there has been considerable interest in their properties. Research has developed in two directions: (i) find properties that transversal matroids have with a view toward characterizing transversal matroids within the class of all matroids, (ii) given a transversal matroid find properties of the presentations of it with a view toward characterizing these presentations. Research of type (i) includes Brualdi and Scrimger [4], Mason [10], and Mirsky and Perfect [12], while research along the lines of (ii) includes Bondy and Welsh [1], Bondy [2], and Las Vergnas [9].

Our investigations are concerned with both properties of transversal matroids and their presentations. We give an algorithm for finding a presentation of a transversal matroid. Since, as we prove, the complements

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of the sets in a presentation are flats of the matroid, the presentation found in the algorithm turns out to be a maximal presentation (the sets can not be enlarged and remain a presentation). It is an immediate consequence of our approach that any maximal presentation must be the one found above and thus that there is only one maximal presentation (Mason [10], Bondy [2]). Thus we obtain an explicit description of the maximal presentation of a transversal matroid. We then obtain a characterization of transversal matroids which can be viewed as an improvement of Mason's characterization [10, 11]. From this characterization we can then characterize all presentations of a transversal matroid. We also obtain a second characterization of the presentations of a transversal matroid.

While there can only be one maximal presentation of a transversal matroid, there may be more than one minimal presentation. Bondy and Welsh [1] and Las Vergnas [9] have demonstrated that the sets in any minimal presentation are cocircuits of the matroid. Even though a minimal presentation need not be unique, Bondy [2] has established the invariance of the cardinalities of the cocircuits in a minimal presentation. Using the fact that the complements of the sets in a presentation are flats, we actually evaluate these cardinalities from which Bondy's result follows immediately.

We close with a discussion of some natural questions which turn out, however, to be answerable in the negative.

2. General Background

Let E be a finite set. A *matroid* $\mathbf{M}$ on E is a non-empty collection of subsets of E, called *independent* sets, satisfying

(2.1) $$A \in \mathbf{M},\ A' \subseteq A \text{ imply } A' \in \mathbf{M}.$$

(2.2) $$A, B \in \mathbf{M},\ |A| + 1 = |B| \text{ imply there exists } x \in B \backslash A \text{ with } A \cup x^1 \in \mathbf{M}.$$

(The cardinality of a set Y is denoted by $|Y|$.)

Subsets of E which are not members of $\mathbf{M}$ are called *dependent sets*. The first systematic investigation of matroids was carried out by Whitney [14], while most of the deep results have been proved by Tutte [13]. For an exposition of the subject we refer the reader to Crapo and Rota [6], where a matroid is termed a pregeometry. Matroids arise in mathematics in many important ways; these include linear spaces, graph theory, geometry, the theory of partitions, and transversal theory.

[1] For simplicity of notation, the set $\{x\}$ is usually denoted by x.

Let $\mathbf{M}$ be a matroid on E. A *basis* of $\mathbf{M}$ is a maximal independent set, that is, a set in $\mathbf{M}$ which is maximal with respect to set-theoretic inclusion. Clearly any independent set can be extended to a basis. It is well known [14] (see also [3]) that all bases of $\mathbf{M}$ have the same cardinality; this common cardinality is called the *rank* of the matroid $\mathbf{M}$. If $F \subseteq E$, then $\mathbf{M}_F = \{A \subseteq F : A \in \mathbf{M}\}$ is a matroid on $\mathbf{M}$, called the *restriction* of $\mathbf{M}$ to F. By above all bases of $\mathbf{M}_F$ have the same cardinality, called the *rank* of F and denoted by $r(F)$. Thus the rank of $\mathbf{M}$ is $r(E)$. Note that $r(F \cup x) \leqslant r(F) + 1$ for any $x \in E$.

A *circuit* of the matroid $\mathbf{M}$ on E is a minimal dependent set. Thus $r(C) = |C| - 1$ and $C \backslash x \in \mathbf{M}$ for all $x \in C$. Axioms for a matroid can be given in terms of its circuits. If $F \subseteq E$ then F is a *flat* of $\mathbf{M}$ provided for all $x \in E \backslash F$ there is no circuit C with $x \in C \subseteq F \cup x$; equivalently F is a flat provided $r(F \cup x) = r(F) + 1$ for all $x \notin F$. For $A \subseteq E$, the *span* of A, $\mathrm{sp}(A)$, is the smallest flat containing A. This is a well-defined concept since the intersection of flats is again a flat. If F is a flat with $r(F) = k$, then F is a flat of rank k or a *k-flat*. If $k = r - 1$ where r is the rank of $\mathbf{M}$, then F is called a *hyperplane*. The collection of flats of $\mathbf{M}$ form a lattice where the meet of two flats F_1 and F_2 is $F_1 \cap F_2$ and the join is $\mathrm{sp}(F_1 \cup F_2)$. In case $\mathbf{M}$ has no circuits of cardinality 2 or less (in which case $\mathbf{M}$ is called a geometry [6]), the lattice of flats is a geometric lattice.

An important concept in matroid theory is that of duality. If $\mathbf{M}$ is a matroid on the set E and $\mathbf{M}^* = \{A \subseteq E : E \backslash A \text{ contains a basis of } \mathbf{M}\}$, then it is well known [14] that $\mathbf{M}^*$ is a matroid on E, called the *dual matroid* of $\mathbf{M}$. The bases of $\mathbf{M}^*$ are the complements of the bases of $\mathbf{M}$. The circuits of $\mathbf{M}^*$ are the *cocircuits* of $\mathbf{M}$; they are those subsets of E which are minimal with respect to the property of intersecting every basis of $\mathbf{M}$ in a non-empty set. It is easy to verify that C is a cocircuit of $\mathbf{M}$ if and only if $E \backslash C$ is a *hyperplane* of $\mathbf{M}$. From this it follows that a circuit and cocircuit cannot have exactly one element in common.

A *loop* of a matroid $\mathbf{M}$ on E is an element x such that $\{x\}$ is a circuit. A *coloop* of $\mathbf{M}$ is an element x such that x is a member of every basis of $\mathbf{M}$. Thus x is a coloop of $\mathbf{M}$ if and only if x is a loop of $\mathbf{M}^*$. If $A \subseteq E$, then we sometimes refer to the coloops of $\mathbf{M}_A$ as coloops of A. We say A or $\mathbf{M}_A$ is *coloop-free* provided $\mathbf{M}_A$ has no coloops. To say A is coloop-free is equivalent to saying that A is the union of circuits of $\mathbf{M}$ or that given $x \in A$ there is a basis of $\mathbf{M}_A$ not containing x. Note that $D \subseteq A$ is a set of coloops of A if and only if $r(A) = r(A \backslash D) + |D|$.

Let E be a set and $\mathfrak{A} = (A_1, A_2, \ldots, A_n)$ a family of (not necessarily distinct) subsets of E. A *partial transversal of* $\mathfrak{A}$ is a set T such that there is an injection $\sigma : T \to \{1, 2, \ldots, n\}$ with $x \in A_{\sigma(x)}$ $(x \in T)$. The injection σ need not be unique. A *transversal* of $\mathfrak{A}$ is a partial transversal T with $|T| = n$

(the σ above can be chosen to be a bijection). The theorem of P. Hall [8] asserts: $\mathfrak{A}$ *has a transversal if and only if*

$$\left| \bigcup_{i \in K} A_i \right| \geqslant | K | \qquad (K \subseteq \{1, 2, ..., n\}).$$

It has been proved by Edmonds and Fulkerson [7] that the collection of partial transversals of $\mathfrak{A}$ form a matroid on E denoted by $\mathbf{M}(\mathfrak{A})$ or $\mathbf{M}(A_1, A_2, ..., A_n)$. Thus if the maximum cardinality of a partial transversal of $\mathfrak{A}$ is k, then every partial transversal of $\mathfrak{A}$ can be enlarged to a partial transversal of cardinality k. By set-element duality if $(A_i : i \in J)$ has a transversal there exists $K \subseteq \{1, ..., n\}$ with $J \subseteq K$ and $| K | = k$ such that $(A_i : i \in K)$ has a transversal. If $\mathbf{M}$ is a matroid on E such that for some family $\mathfrak{A}$ of sets $\mathbf{M} = \mathbf{M}(\mathfrak{A})$, then $\mathbf{M}$ is called a *transversal matroid* and $\mathfrak{A}$ is a *presentation* of M. There are in general many presentations for a given transversal matroid $\mathbf{M}$. One class of transversal matroids are matroids $\mathbf{M}$ with $\mathbf{M} = \mathscr{P}_r(E)$, the collection of all subsets of E of cardinality at most r. A presentation is $(E, E, ..., E)$(r times).

Finally, if $\mathfrak{A} = (A_i : i \in I)$ is a family, indexed by I, of subsets of a set E, then the *cardinality of the family* $\mathfrak{A}$ is defined to be the cardinality, $| I |$, of the index set I. For each $i \in I$, A_i is a *member of the family* $\mathfrak{A}$. Two members of $\mathfrak{A}$ may be equal. If $F \subseteq E$, we say that F has *multiplicity* $m(\mathfrak{A}; F) = m \geqslant 0$ in $\mathfrak{A}$ if

$$|\{i \in I : A_i = F\}| = m.$$

If $\mathfrak{B} = (B_j : j \in J)$ is another family of subsets of E, then it will be convenient for us to regard $\mathfrak{B}$ as a *subfamily* of $\mathfrak{A}$ if $m(\mathfrak{B}; F) \leqslant m(\mathfrak{A}; F)$ for each $F \subseteq E$, and to regard $\mathfrak{B}$ and $\mathfrak{A}$ as *equal families* if $m(\mathfrak{B}; F) = m(\mathfrak{A}; F)$ for each $F \subseteq E$.

3. Presentations of Transversal Matroids

Except for Theorems 3.1 and 3.6 the theorems in this section are known. The first, which is new, says that the complements of the sets in any presentation of a transversal matroid must be flats of the matroid.

Theorem 3.1. *If* $\mathbf{M}$ *is the transversal matroid* $\mathbf{M}(A_1, A_2, ..., A_n)$, *then* $E \backslash A_i$ *is a flat of* $\mathbf{M}$ $(1 \leqslant i \leqslant n)$.

Proof. It suffices to prove that $E \backslash A_1$ is a flat of $\mathbf{M}$. If $A_1 = \varnothing$, this is clear. Thus assume $A_1 \neq \varnothing$, and let B be a basis of $\mathbf{M}_{E \backslash A_1}$ which equals

$\mathbf{M}(A_2 \backslash A_1, \ldots, A_n \backslash A_1)$ and let $x \in A_1$. Since B is a partial transversal of $(A_2 \backslash A_1, \ldots, A_n \backslash A_1)$, $B \cup x$ is a partial transversal of $(A_1, A_2, \ldots, A_n)$ so that $B \cup x \in \mathbf{M}$. Thus for all $x \in A_1$, x is not in the span of $E \backslash A_1$ so that $sp(E \backslash A_1) = E \backslash A_1$ and $E \backslash A_1$ is a flat.

The next theorem says, in particular, that we need only consider presentations consisting of r sets of a transversal matroid of rank r.

THEOREM 3.2 (Brualdi and Scrimger [4], Mason [10], Brualdi and Mason [5]). *Let M be the transversal matroid $\mathbf{M}(A_1, A_2, \ldots, A_n)$ of rank $r \leqslant n$ where $A_i \neq \varnothing$ $(1 \leqslant i \leqslant n)$. If $(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$ has a transversal where $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$, then $\mathbf{M} = \mathbf{M}(A_{i_1}, A_{i_2}, \ldots, A_{i_r})$. If $\mathbf{M}$ is a coloop-free matroid, then $r = n$.*

Thus while a transversal matroid of rank r may be presented by more than r non-empty sets, a coloop-free transversal matroid can not. As a consequence of the theorem we have

COROLLARY 3.3. *Let $\mathbf{M}$ be the transversal matroid $\mathbf{M}(A_1, A_2, \ldots, A_n)$. Let $A \subseteq E$ where $\mathbf{M}_A$ is a coloop-free matroid of rank k. If*

$$K = \{i : 1 \leqslant i \leqslant n, A_i \cap A \neq \varnothing\}, \text{ then } |K| = k.$$

This corollary follows from Theorem 3.2, since

$$M_A = M(A_1 \cap A, \ldots, A_n \cap A).$$

The next theorem is due to Bondy and Welsh. It describes the circumstances under which a presentation of a transversal matroid can be "enlarged" and remain a presentation. Their proof of this theorem depends on a reduction theorem.

THEOREM 3.4 (Bondy and Welsh [1]). *Let $\mathbf{M}$ be the transversal matroid $\mathbf{M}(A_1, A_2, \ldots, A_n)$ of rank $r \leqslant n$ on E. Let $X \subseteq E \backslash A_1$. Then*

$$\mathbf{M} = \mathbf{M}(A_1 \cup X, A_2, \ldots, A_n)$$

if and only if X is a set of coloops of $\mathbf{M}_{E \backslash A_1} = \mathbf{M}(A_2 \backslash A_1, \ldots, A_n \backslash A_1)$.

If $\mathbf{M}$ is a transversal matroid of rank r with $\mathbf{M} = \mathbf{M}(M_1, M_2, \ldots, M_r)$, then $(M_1, M_2, \ldots, M_r)$ is a *maximal presentation* of $\mathbf{M}$ if

$$\mathbf{M} = \mathbf{M}(N_1, N_2, \ldots, N_r), \; M_i \subseteq N_i (1 \leqslant i \leqslant r)$$

imply $N_i = M_i$ $(1 \leqslant i \leqslant r)$. By Theorem 3.2 a presentation of a transversal matroid or rank r with no coloops can only consist of r non-empty sets. If the matroid has coloops, these coloops can be contained in an unlimited number of sets in a presentation; in this case there can be no truly maximal presentation.

COROLLARY 3.5. *If* $(M_1, M_2, ..., M_r)$ *is a maximal presentation of the transversal matroid* $\mathbf{M}$ *of rank* r, *then* $E\backslash M_i$ *is a coloop-free flat of* $\mathbf{M}$ $(1 \leqslant i \leqslant r)$.

This is a direct consequence of Theorems 3.1 and 3.4.

Theorem 3.4 describes the conditions under which an element may be added to a set in a presentation of a transversal matroid leaving the matroid unchanged. The next theorem is complementary to this; it describes when an element may be deleted without changing the matroid.

THEOREM 3.6. *Let* $\mathbf{M}$ *be the transversal matroid* $\mathbf{M}(A_1, A_2, ..., A_n)$ *of rank* r. *Let the matroid* $\mathbf{M}_{E\backslash A_1}$ *have rank* k (k *cannot exceed* $r - 1$ *if* $A_1 \neq \varnothing$). *Let* $x \in A_1$, *and suppose* $x \in A_i (2 \leqslant i \leqslant t)$, $x \notin A_i (t + 1 \leqslant i \leqslant n)$. *Then* $\mathbf{M} = \mathbf{M}(A_1\backslash x, A_2, ..., A_n)$ *if and only if for some* p *with* $2 \leqslant p \leqslant t$, $\mathbf{M}(A_i\backslash A_1 : 2 \leqslant i \leqslant n, i \neq p)$ *has rank* k.

Proof. Suppose for some p with $2 \leqslant p \leqslant t$ that $\mathbf{M}(A_i\backslash A_1 : 2 \leqslant i \leqslant n, i \neq p)$ has rank k. Then by Theorem 3.2 $\mathbf{M}(A_i\backslash A_1 : 2 \leqslant i \leqslant n) = \mathbf{M}(A_i\backslash A_1 : 2 \leqslant i \leqslant n, i \neq p)$. Hence, since $x \in A_p$, $\mathbf{M}(A_2\backslash\{A_1\backslash x\}, ..., A_n\backslash\{A_1\backslash x\})$ has rank $k + 1$ and thus has x as a coloop. From Theorem 3.4 we conclude that $\mathbf{M}(A_1\backslash x, A_2, ..., A_n) = \mathbf{M}(A_1, A_2, ..., A_n)$.

Conversely, suppose $\mathbf{M}(A_1\backslash x, A_2, ..., A_n) = \mathbf{M}(A_1, A_2, ..., A_n)$. Then, by Theorem 3.4, x is a coloop of $\mathbf{M}_{E\backslash\{A_1\backslash x\}} = \mathbf{M}(A_2\backslash\{A_1\backslash x\}, ..., A_n\backslash\{A_1\backslash x\})$. Thus, since $\mathbf{M}_{E\backslash A_1}$ has rank k, $M_{E\backslash\{A_1\backslash x\}}$ has rank $k + 1$. But this means that, for some p with $2 \leqslant p \leqslant t$, $\mathbf{M}(A_i\backslash A_1 : 2 \leqslant i \leqslant n, i \neq p)$ has rank k.

THEOREM 3.7 (Bondy and Welsh [1], Las Vergnas [9]). *Let* $\mathbf{M}$ *be the transversal matroid* $\mathbf{M}(A_1, A_2, ..., A_r)$ *of rank* r. *If* A_1 *is not a cocircuit of* $\mathbf{M}$, *there exists* $x \in A_1$ *such that* $\mathbf{M} = \mathbf{M}(A_1\backslash x, A_2, ..., A_r)$. *Thus there exist cocircuits* $C_i \subseteq A_i (1 \leqslant i \leqslant r)$, *necessarily distinct, such that* $\mathbf{M} = \mathbf{M}(C_1, C_2, ..., C_r)$.

This theorem can be proved by use of Theorem 3.6.

If $\mathbf{M}$ is a transversal matroid of rank r and $C_1, ..., C_r$ are cocircuits such that $\mathbf{M} = \mathbf{M}(C_1, ..., C_r)$, then $(C_1, ..., C_r)$ is a *minimal presentation* in the sense that no element can be removed from any C_1 without altering the matroid presented. By Theorems 3.2 and 3.7 every minimal presentation consists of r non-empty sets and these sets are distinct cocircuits.

4. An Algorithm and Characterizations

In this section we present the main results of our investigations. We give an algorithm for obtaining the maximal presentation of a transversal matroid. From this we derive a characterization of transversal matroids and of their presentations. The first theorem furnishes the first step for the algorithm.

THEOREM 4.1. *Let* $\mathbf{M}$ *be the transversal matroid* $\mathbf{M}(A_1, A_2, \ldots, A_r)$ *of rank* r. *Let* $F_1, \ldots, F_t$ *be the distinct coloop-free hyperplanes of* $\mathbf{M}$. *Then, after renumbering,* $A_i = E \backslash F_i (1 \leqslant i \leqslant t)$, *while*

$$A_j \neq E \backslash F_i (t+1 \leqslant j \leqslant r, 1 \leqslant i \leqslant t).$$

If $(M_1, M_2, \ldots, M_r)$ *is a maximal presentation of* $\mathbf{M}$ *and* F *is a hyperplane different from* $F_1, \ldots, F_t$, *then* $M_i \neq E \backslash F (1 \leqslant i \leqslant r)$.

Proof. Let $(M_1, M_2, \ldots, M_r)$ be a maximal presentation of $\mathbf{M}$. Let F_1 be a coloop-free hyperplane of $\mathbf{M}$. Since $E \backslash F_1$ is a cocircuit, if we can prove that $M_i = E \backslash F_1$ for some i, $1 \leqslant i \leqslant r$, the result holds true for any presentation.

By Corollary 3.3, since F_1 has rank $r-1$, $\{i : 1 \leqslant i \leqslant r, F_1 \cap M_i \neq \varnothing\}$ is a set of cardinality $r-1$. Hence after renumbering we may assume $F_1 \cap M_1 = \varnothing$ so that $M_1 \subseteq E \backslash F_1$. But $E \backslash F_1$ is a cocircuit and M_1 must contain a cocircuit. Thus $M_1 = E \backslash F_1$. Suppose for some $i \neq 1$, $M_i = E \backslash F_1$; then $M_1 = M_i$. Hence for any $x \in M_1$, $(M_1 \backslash x, M_2, \ldots, M_r)$ is a presentation of $\mathbf{M}$. But this is a contradiction, for M_1 is a cocircuit and thus $M_1 \backslash x$ cannot contain a cocircuit.

If F is a hyperplane which is not coloop-free it follows from Theorem 3.4 that $M_i \neq E \backslash F$ for all i, $1 \leqslant i \leqslant r$.

We now give an algorithm for obtaining from the matroid $\mathbf{M}$ on E a family $\mathscr{F} = (F_1, F_2, \ldots, F_n)$ of flats. Theorem 4.2 demonstrates the significance of $\mathscr{F}$ when $\mathbf{M}$ is a transversal matroid.

THE ALGORITHM. (1) *Construct the family* $\mathscr{F}^{(1)}$ *consisting of the coloop-free hyperplanes each with multiplicity* 1.

(2) *Construct a family* $\mathscr{F}^{(2)}$ *which is to include* $\mathscr{F}^{(1)}$ *as a subfamily and certain of the coloop-free* $(r-2)$*-flats with multiplicity* 1 *or* 2. *Let* F *be a coloop-free* $(r-2)$*-flat contained in* p *coloop-free hyperplanes. If* $p \geqslant 2$, F *is not a member of* $\mathscr{F}^{(2)}$; *otherwise* F *is a member of* $\mathscr{F}^{(2)}$ *with multiplicity* $2-p$.

...

(k) Construct a family $\mathscr{F}^{(k)}$ which is to include $\mathscr{F}^{(k-1)}$ as a subfamily and certain of the coloop-free $(r - k)$-flats with multiplicity $m(1 \leqslant m \leqslant k)$. Let F be a coloop-free $(r - k)$-flat suppose F is contained in p members of $\mathscr{F}^{(k-1)}$. If $p \geqslant k$, F is not a member of $\mathscr{F}^{(k)}$; otherwise F is a member of $\mathscr{F}^{(k)}$ with multiplicity $k - p$.

...

(r) Construct a family $\mathscr{F}^{(r)}$ which is to include $\mathscr{F}^{(r-1)}$ as a subfamily and the span of the empty set, sp ϕ, with multiplicity m, $0 \leqslant m \leqslant r$. Suppose $\mathscr{F}^{(r-1)}$ has p members counting repetitions. If $p \geqslant r$, then sp ϕ is not a member of $\mathscr{F}^{(r)}$; otherwise sp ϕ is a member of $\mathscr{F}^{(r)}$ with multiplicity $r - p$.

Let $\mathscr{F} = \mathscr{F}^{(r)}$. Then, counting repetitions, $\mathscr{F}$ is a family of coloop-free flats of cardinality $n \geqslant r$.

THEOREM 4.2. *If* $\mathbf{M}$ *is a transversal matroid of rank* r, *then the maximal presentation is unique. Indeed if* $\mathscr{F} = (F_1, F_2, ..., F_n)$ *is the family obtained in the algorithm, then* $n = r$ *and* $(E\backslash F_1, ..., E\backslash F_r)$ *is the maximal presentation of* $\mathbf{M}$.

Proof. Let $\mathscr{M} = (M_1, M_2, ..., M_r)$ be any maximal presentation of $\mathbf{M}$. By Corollary 3.5 the complements of the M_i are coloop-free flats of rank less than r. We prove, by induction on k, that the members of the family $\mathscr{F}^{(k)}$ are those complements of members of $\mathscr{M}$ that have rank at least $r - k$. The statement in the theorem is equivalent to this assertion for $k = r$. For $k = 1$ we need to verify then the members of $\mathscr{F}^{(1)}$ are precisely the complements of the members of $\mathscr{M}$ that are coloop-free hyperplanes. This is the assertion of Theorem 4.1. Assume now the above assertion is true for $k - 1 \geqslant 1$; we prove it true for k.

Let F be a coloop-free flat of rank $r - k$. Let $I = \{i : 1 \leqslant i \leqslant r, F \cap M_i \neq \varnothing\}$. Since F is a coloop-free $(r - k)$-flat, we conclude from Corollary 3.3 that $|I| = r - k$. Thus $F \subseteq E\backslash M_i$ for all $i \notin I$. This means F is contained in the complement of exactly k members of $\mathscr{M}$. Let p be the number of members of the family $\mathscr{F}^{(k-1)}$ which contain F. By the inductive assumption there are then exactly p members of $\mathscr{M}$ whose complements contain, but are not equal to, F. Hence there are exactly $k - p$ members of $\mathscr{M}$ whose complement equals F. By the algorithm there are exactly $k - p$ members of $\mathscr{F}^{(k)}$ that equal F. Since this is true for all coloop-free $(r - k)$-flats, this proves the assertion for k. Hence by induction $\mathscr{M} = (E\backslash F_1, ..., E\backslash F_r)$ and, in particular, the maximal presentation is unique.

Contained in the proof of the preceding theorem is a proof of the following corollary:

COROLLARY 4.3. *If F is a coloop-free $(r-k)$-flat, $2 \leqslant k \leqslant r$, of a transversal matroid of rank r, then there are at most k members of $\mathscr{F}^{(k-1)}$ which contain F.*

This corollary furnishes a necessary condition for a matroid to be a transversal matroid. The example for Query 5.2 in Section 5 shows that it is not sufficient.

Before obtaining necessary and sufficient conditions for a matroid to be a transversal matroid, we prove the following lemma:

LEMMA 4.4. *Let $\mathbf{M}$ be a matroid of rank r on E, and let $(A_1, A_2, \ldots, A_r)$ be an arbitrary family of subsets of E. Then $\mathbf{M} \subseteq \mathbf{M}(A_1, A_2, \ldots, A_r)$ if and only if*

$$r\left(\bigcap_{i\in I} \{E \backslash A_i\}\right) \leqslant r - |I| \qquad (I \subseteq \{1, 2, \ldots, r\}). \tag{4.1}$$

Proof. Suppose (4.1) holds. Let B be a basis of $\mathbf{M}$; $|B| = r$. Suppose for some $J \subseteq \{1, 2, \ldots, r\}$,

$$\left|\left\{\bigcup_{i\in J} A_i\right\} \cap B\right| < |J|.$$

Then

$$\left|B \cap \left(E \Big\backslash \bigcup_{i\in J} A_i\right)\right| \geqslant r - |J| + 1,$$

so that

$$r\left(\bigcap_{i\in J} \{E \backslash A_i\}\right) = r\left(E \Big\backslash \bigcup_{i\in J} A_i\right) \geqslant r - |J| + 1.$$

This is a contradiction. Thus for all $I \subseteq \{1, 2, \ldots, r\}$

$$\left|\bigcup_{i\in I} \{A_i \cap B\}\right| \geqslant |I|,$$

and by Hall's theorem B contains a transversal of $(A_1, A_2, \ldots, A_r)$. Since $|B| = r$, B is a transversal of $(A_1, A_2, \ldots, A_r)$. Hence $\mathbf{M} \subseteq \mathbf{M}(A_1, A_2, \ldots, A_r)$.

Now suppose $\mathbf{M} \subseteq \mathbf{M}(A_1, A_2, \ldots, A_r)$. Thus $\mathbf{M}(A_1, A_2, \ldots, A_r)$ has rank r and every basis of $\mathbf{M}$ is a transversal of $(A_1, A_2, \ldots, A_r)$. By Hall's theorem we conclude that

$$\left|\left\{\bigcup_{i\in I} A_i\right\} \cap B\right| \geqslant |I| \qquad (I \subseteq \{1, 2, \ldots, r\})$$

for every basis B of $\mathbf{M}$. Thus

$$\left| B \cap \left(E \Big\backslash \bigcup_{i \in J} A_i\right)\right| \leqslant r - |J| \qquad (J \subseteq \{1, 2,..., r\})$$

for every basis B of $\mathbf{M}$. We conclude that

$$r\left(E \Big\backslash \bigcup_{i \in J} A_i\right) \leqslant r - |J| \qquad (J \subseteq \{1, 2,..., r\}).$$

THEOREM 4.5. *Let* $\mathbf{M}$ *be a matroid of rank* r *on* E *and let* $\mathscr{F} = (F_1, F_2, ..., F_n)$ *be the family of sets obtained from the algorithm applied to* $\mathbf{M}$. *Let* $M_i = E \backslash F_i (1 \leqslant i \leqslant n)$. *Then* $\mathbf{M} = \mathbf{M}(M_1, M_2, ..., M_n)$ *if and only if*

$$r\left(\bigcap_{i \in I} F_i\right) \leqslant r - |I| \qquad (I \subseteq \{1, 2,..., n\}). \tag{4.2}$$

Proof. If $\mathbf{M} = \mathbf{M}(M_1, M_2, ..., M_n)$, then $\mathbf{M}$ is a transversal matroid and, by Theorem 4.2, $r = n$ and $(M_1, M_2, ..., M_r)$ is the maximal presentation of M. By Lemma 4.4, (4.2) is satisfied.

Conversely, suppose (4.2) is satisfied. By taking $I = \{1, 2,..., n\}$ we see that $n \leqslant r$. But from the algorithm we know $r \leqslant n$; hence $n = r$. Since (4.2) is valid, we know from Lemma 4.4 that $\mathbf{M} \subseteq \mathbf{M}(M_1, M_2, ..., M_r)$. Thus we need only show that $\mathbf{M}(M_1, M_2, ..., M_r) \subseteq \mathbf{M}$ or that every transversal of $(M_1, M_2, ..., M_r)$ is in $\mathbf{M}$.

Suppose T is a transversal of $(M_1, M_2, ..., M_r)$ with $T \notin \mathbf{M}$. Let A be the union of all circuits in T and let F equal the span of A in M. Then for some k, F is a coloop-free flat of rank $r - k$ in $\mathbf{M}$. Since A is the union of circuits we have

$$r - k = r(A) < |A| \tag{4.3}$$

and thus that

$$|T \backslash A| \leqslant k - 1. \tag{4.4}$$

Since $T \notin M$, $k \geqslant 1$. Let $J = \{i \in I : F \subseteq F_i = E \backslash M_i\}$. Then by (4.2)

$$r - k = r(F) \leqslant r\left(\bigcap_{i \in J} F_i\right) \leqslant r - |J|$$

and thus $|J| \leqslant k$. But from the algorithm we conclude that $|J| \geqslant k$; hence $|J| = k$. Summarizing, we have a set $J \subseteq \{1, 2,..., r\}$ with $|J| = k$ such that for all $i \in J, F \cap M_i = \varnothing$. If r' denotes the rank function of the matroid $\mathbf{M}' = \mathbf{M}(M_1, M_2, ..., M_r)$, this implies that $r'(F) \leqslant r - k$. Hence using (4.3) and (4.4), we have

$$\begin{aligned} r'(T) &\leqslant r'(A) + r'(T \backslash A) \\ &\leqslant (r-k) + |T \backslash A| \\ &\leqslant (r-k) + (k-1) \\ &= r - 1. \end{aligned}$$

This contradicts the assumption that T is a transversal of $(M_1, M_2, ..., M_r)$ and thus that $r'(T) = r$. This completes the proof of the theorem.

From Theorems 4.2 and 4.5 we now obtain a necessary and sufficient condition for a matroid to be a transversal matroid.

THEOREM 4.6. *Let* $\mathbf{M}$ *be a matroid of rank* r *on* E *and let*

$$\mathscr{F} = (F_1, F_2, ..., F_n)$$

be the family of sets obtained by applying the algorithm to $\mathbf{M}$. *Then* $\mathbf{M}$ *is a transversal matroid if and only if*

$$r\left(\bigcap_{i \in I} F_i\right) \leqslant r - |I| \qquad (I \subseteq \{1, 2, ..., n\}). \tag{4.5}$$

Proof. From Theorem 4.2 we conclude that $\mathbf{M}$ is a transversal matroid if and only if $\mathbf{M} = \mathbf{M}(E \backslash F_1, ..., E \backslash F_n)$. From Theorem 4.5 we have that $\mathbf{M} = \mathbf{M}(E \backslash F_1, ..., E \backslash F_n)$ if and only if (4.5) is satisfied.

We now obtain necessary and sufficient conditions for a family of sets to be a presentation of a given transversal matroid.

THEOREM 4.7. *Let* $\mathbf{M}$ *be a transversal matroid of rank* r *on* E *with maximal presentation* $(M_1, M_2, ..., M_r)$. *Let* $(A_1, A_2, ..., A_r)$ *be a family of subsets of* E. *Then* $\mathbf{M} = \mathbf{M}(A_1, A_2, ..., A_r)$ *if and only if*

(4.6) *there exists a permutation* $(i_1, i_2, ..., i_r)$ *of* $\{1, 2, ..., r\}$ *such that* $A_k \subseteq M_{i_k} (k = 1, 2, ..., r)$,

$$r\left(\bigcap_{i \in I} \{E \backslash A_i\}\right) \leqslant r - |I| \qquad (I \subseteq \{1, 2, ..., r\}). \tag{4.7}$$

Proof. If $\mathbf{M} = \mathbf{M}(A_1, A_2, ..., A_r)$ then by Lemma 4.4, (4.7) is satisfied. Since $(A_1, A_2, ..., A_r)$ can be enlarged to a maximal presentation and since by Theorem 4.2 the maximal presentation is unique, (4.6) is satisfied.

Conversely suppose (4.6) and (4.7) are fulfilled. Then, by Lemma 4.4, $\mathbf{M} \subseteq \mathbf{M}(A_1, A_2, ..., A_r)$. But from (4.6) we conclude

$$\mathbf{M}(A_1, A_2, ..., A_r) \subseteq \mathbf{M}(M_1, M_2, ..., M_r) = M.$$

Hence $\mathbf{M} = \mathbf{M}(A_1, A_2, ..., A_r)$.

Theorem 4.7 characterizes the presentations of a given transversal matroid. We now obtain some information about the members of these presentations. No use is made here of the algorithm; we do, however, make use of the uniqueness of the maximal presentation.

LEMMA 4.8. *Let* $\mathbf{M}$ *be a matroid on* E *and let* $D \subseteq A$. *Then*

$$| D | \leqslant | A | - (r(E\backslash D) - r(E\backslash A)).$$

Equality occurs if and only if $A\backslash D$ *consists entirely of coloops of* $\mathbf{M}_{E\backslash D}$.

Proof.
$$| A\backslash D | = |\{E\backslash D\}\backslash\{E\backslash A\}| \geqslant r(E\backslash D) - r(E\backslash A),$$

since $D \subseteq A$. Equality means

$$r(E\backslash D) = r(E\backslash A) + | A\backslash D |,$$

and this is equivalent to $A\backslash D$ being a set of coloops of $\mathbf{M}_{E\backslash D}$, since $E\backslash D = \{E\backslash A\} \cup \{A\backslash D\}$.

COROLLARY 4.9. *Let* $\mathbf{M}$ *be a transversal matroid of rank* r *with maximal presentation* $(M_1, M_2, ..., M_r)$. *Suppose* $\mathbf{M} = \mathbf{M}(A_1, A_2, ..., A_r)$ *where* $A_i \subseteq M_i$ $(1 \leqslant i \leqslant r)$. *Then*

$$(4.8) \qquad | A_i | = | M_i | - (r(E\backslash A_i) - r(E\backslash M_i)) \qquad (1 \leqslant i \leqslant r).$$

Proof. Since both $(A_1, A_2, ..., A_r)$ and $(M_1, M_2, ..., M_r)$ are presentations and $A_i \subseteq M_i (1 \leqslant i \leqslant r)$, $(M_1, A_2, ..., A_r)$ is also a presentation of $\mathbf{M}$. By Theorem 3.4, $M_1\backslash A_1$ consists entirely of coloops of $\mathbf{M}_{E\backslash A_1}$. The result now follows from Lemma 4.8.

THEOREM 4.10. *Let* $\mathbf{M}$ *be a transversal matroid of rank* r *with maximal presentation* $(M_1, M_2, ..., M_r)$. *Suppose* $\mathbf{M} = \mathbf{M}(A_1, A_2, ..., A_r)$ *where* $A_i \subseteq M_i$ *and* $r(E\backslash A_i) = k_i (1 \leqslant i \leqslant r)$. *Then* $| A_i |$ *is the maximum car-*

dinality of all subsets of M_i *whose complement has rank* $k_i (1 \leqslant i \leqslant r)$. *In particular if also* $\mathbf{M} = \mathbf{M}(A_1', A_2', ..., A_r')$ *where* $A_i' \subseteq M_i$ *and*

$$r(E|A_i') = k_i (1 \leqslant i \leqslant r),$$

then $| A_i | = | A_i' | (1 \leqslant i \leqslant r)$.

Proof. By Corollary 4.9,

$$| A_i | = | M_i | - \{k_i - r(E \backslash M_i\}.$$

By Lemma 4.8, if $D \subseteq M_i$ with $r(E \backslash D) = k_i \ (1 \leqslant i \leqslant r)$, then

$$| D | \leqslant | M_i | - (k_i - r(E \backslash M_i))(1 \leqslant i \leqslant r).$$

The first assertion now follows; the second assertion follows from the first.

As previously remarked, while the maximal presentation of a transversal matroid is unique, there may be more than one minimal presentation. By Theorem 3.7, the members of a minimal presentation must be cocircuits. The following corollary specifies their cardinalities.

COROLLARY 4.11. *Let* $\mathbf{M}$ *be a transversal matroid with maximal presentation* $(M_1, M_2, ..., M_r)$. *Suppose* $\mathbf{M} = \mathbf{M}(C_1, C_2, ..., C_r)$ *where* $C_i \subseteq M_i$ *with* C_i *a cocircuit* $(1 \leqslant i \leqslant r)$. *Then*

$$| C_i | = | M_i | - ((r - 1) - r(E \backslash M_i)) \quad (1 \leqslant i \leqslant r).$$

This follows from Corollary 4.9, since $r(E \backslash C_i) = r - 1 \ (1 \leqslant i \leqslant r)$.

COROLLARY 4.12 (Bondy [2]). *Let* $\mathbf{M}$ *be a transversal matroid of rank* r *with maximal presentation* $(M_1, M_2, ..., M_r)$. *Suppose*

$$M = M(C_1, C_2, ..., C_r)$$

where $C_i \subseteq M_i$ *with* C_i *a cocircuit* $(1 \leqslant i \leqslant r)$. *Then* $| C_i |$ *is the maximum of the cardinalities of the cocircuits contained in* $M_i (1 \leqslant i \leqslant r)$.

This is a direct consequence of Theorem 4.10, since $k_i = r - 1 (1 \leqslant i \leqslant r)$ and cocircuits are complements of hyperplanes.

LEMMA 4.13. *Let* $\mathbf{M}$ *be a transversal matroid with maximal presentation* $(M_1, M_2, ..., M_r)$. *Let* $A \subseteq M_1 \cap M_2$ *with* $r(E \backslash A) = k < r$. *If for* $i = 1, 2$ $| A |$ *is the maximum cardinality of all subsets of* M_i *whose complements have rank* k, *then* $M_1 = M_2$.

Proof. Suppose A has the properties stated in the lemma. Then, by Lemma 4.8, $| A | \leqslant | M_i | - (k - r(E \backslash M_i))$ for $i = 1, 2$. If equality does not hold, then there is an $x \in M_i \backslash A$ which is not a coloop of $\mathbf{M}_{E \backslash A}$. Hence

$r(E\backslash\{A \cup x\}) = k$, and we contradict the maximality condition on A. Thus equality occurs for $i = 1, 2$. By Lemma 4.8 again, $M_i\backslash A$ consists of coloops of $\mathbf{M}_{E\backslash A}(i = 1, 2)$. Since, by Corollary 4.5, $\mathbf{M}_{E\backslash M_i}$ is coloop-free, $M_i\backslash A$ is the entire set of coloops of $\mathbf{M}_{E\backslash A}(i = 1, 2)$. Thus $M_1\backslash A = M_2\backslash A$ and since $A \subseteq M_1 \cap M_2$, $M_1 = M_2$.

Lemma 4.13 contains as special case the assertion that a maximum cardinality cocircuit in M_i cannot be a maximum cardinality in M_j if $M_i \neq M_j$.

THEOREM 4.14. *Let* $\mathbf{M}$ *be a transversal matroid with maximal presentation* $(M_1, M_2, \ldots, M_r)$. *Suppose* $(A_1, A_2, \ldots, A_r)$ *and* $(B_1, B_2, \ldots, B_r)$ *are presentations of* $\mathbf{M}$ *with* $A_i, B_i \subseteq M_i (1 \leqslant i \leqslant r)$. *If* $A_1 = B_2 = D$, *then* $M_1 = M_2$.

Proof. Let $r(E\backslash D) = k$. Then, by Theorem 4.10, $|D|$ is the maximum cardinality of all subsets of M_i whose complement has rank k $(i = 1, 2)$. By Lemma 4.13, $M_1 = M_2$.

This theorem contains a subtle observation about presentations. It implies that every set which is a member of some presentation of $\mathbf{M}$ is uniquely associated with a *set* in the maximal presentation (there may be more than one member in the maximal presentation equal to the latter set). More precisely, if $(A_1, A_2, \ldots, A_r)$ is a presentation of $\mathbf{M}$ with

$$A_k \subseteq M_k (1 \leqslant k \leqslant r) \qquad \text{and} \quad (i_1, i_2, \ldots, i_r)$$

is a permutation of $\{1, 2, \ldots, r\}$ with $A_k \subseteq M_{i_k} (1 \leqslant k \leqslant r)$, then $M_{i_k} = M_k (1 \leqslant k \leqslant r)$.

Another consequence is that, if two members of some presentation of $\mathbf{M}$ are equal as sets, then $M_i = M_j$ for some $i \neq j$.

We now obtain another necessary and sufficient condition for a family of sets to be a presentation of a given transversal matroid. Before stating this result we prove a lemma:

LEMMA 4.15. *Let* $\mathbf{M}$ *be the transversal matroid* $M(A_1, A_2, \ldots, A_r)$ *of rank* r. *Suppose* $(B_1, B_2, \ldots, B_r)$ *is a family of sets with* $A_i \subseteq B_i (1 \leqslant i \leqslant r)$ *such that*

$$\mathbf{M} = \mathbf{M}(B_1, A_2, \ldots, A_r)$$
$$\mathbf{M} = \mathbf{M}(A_1, B_2, \ldots, A_r)$$
$$\cdots$$
$$\mathbf{M} = \mathbf{M}(A_1, A_2, \ldots, B_r).$$

Then $\mathbf{M} = \mathbf{M}(B_1, B_2, \ldots, B_r)$.

Proof. Let $(M_1, M_2, \ldots, M_r)$ be the maximal presentation of $\mathbf{M}$ where we may assume after renumbering that $A_i \subseteq M_i (1 \leqslant i \leqslant r)$. Since $\mathbf{M} = \mathbf{M}(B_1, A_2, \ldots, A_r)$, we conclude from Theorem 3.4 that $B_1 \backslash A_1$ consists of coloops of $\mathbf{M}_{E \backslash A_1}$; on the other hand, since by Corollary 3.5 $\mathbf{M}_{E \backslash M_1}$ is coloop-free, $M_1 \backslash A_1$ consists of all the coloops of $\mathbf{M}_{E \backslash A_1}$. Thus $B_1 \backslash A_1 \subseteq M_1 \backslash A_1$ or $A_1 \subseteq B_1 \subseteq M_1$. In the same way we prove

$$A_i \subseteq B_i \subseteq M_i (1 \leqslant i \leqslant r).$$

Thus

$$\mathbf{M} \subseteq \mathbf{M}(B_1, B_2, \ldots, B_r) \subseteq \mathbf{M}(M_1, M_2, \ldots, M_r) = \mathbf{M}$$

or

$$\mathbf{M} = \mathbf{M}(B_1, B_2, \ldots, B_r).$$

THEOREM 4.16. *Let* $\mathbf{M}$ *be a transversal matroid of rank r with maximal presentation* $(M_1, M_2, \ldots, M_r)$. *Let* $(A_1, A_2, \ldots, A_r)$ *be a family of sets and set* $\mathbf{M}' = \mathbf{M}(A_1, A_2, \ldots, A_r)$. *Then* $\mathbf{M} = \mathbf{M}'$ *if and only if there exists a permutation* $(i_1, i_2, \ldots, i_r)$ *of* $\{1, 2, \ldots, r\}$ *with* $A_k \subseteq M_{i_k} k = 1, 2, \ldots, r)$ *such that*

$$M_{i_k} \backslash A_k \text{ consists entirely of coloops of } \mathbf{M}_{E \backslash A_k} \quad (k = 1, 2, \ldots, r), \tag{4.9}$$

and

$$\text{the rank of the matroid } \mathbf{M}_{E \backslash A_k} \text{ equals the rank of the matroid } \mathbf{M}'_{E \backslash A_k} \text{ for all } k \text{ such that } A_k \neq M_{i_k} \quad (k = 1, 2, \ldots, r). \tag{4.10}$$

Proof. Suppose $\mathbf{M} = \mathbf{M}(A_1, A_2, \ldots, A_r) = \mathbf{M}'$. Since the maximal presentation is unique, we may assume after renumbering that $A_i \subseteq M_i (1 \leqslant i \leqslant r)$. Thus $\mathbf{M} = \mathbf{M}(M_1, A_2, \ldots, A_r)$. By Theorem 3.4, $M_1 \backslash A_1$ consists of coloops of $\mathbf{M}_{E \backslash A_1}$; similarly $M_i \backslash A_i$ consists of coloops of $\mathbf{M}_{E \backslash A_i} (1 \leqslant i \leqslant r)$. Since $\mathbf{M} = \mathbf{M}'$, $\mathbf{M}_{E \backslash A_i} = \mathbf{M}'_{E \backslash A_i} (1 \leqslant i \leqslant r)$. Thus the conditions in the theorem are necessary.

Suppose the conditions in the theorem are satisfied. After renumbering we may assume $A_i \subseteq M_i$. By Theorem 3.4, $\mathbf{M}' = \mathbf{M}(A_1, A_2, \ldots, A_r) = \mathbf{M}(M_1, A_2, \ldots, A_r)$ if and only if $M_1 \backslash A_1$ consists of coloops of $\mathbf{M}'_{E \backslash A_1}$. Suppose $A_1 \neq M_1$. By (4.9) $M_1 \backslash A_1$ consists of coloops of $\mathbf{M}_{E \backslash A_1}$. But $\mathbf{M}'_{E \backslash A_1} \subseteq \mathbf{M}_{E \backslash A_1}$ with, by (4.10), the rank of $\mathbf{M}'_{E \backslash A_1}$ equal to the rank of $\mathbf{M}_{E \backslash A_1}$. Thus every basis of $\mathbf{M}'_{E \backslash A_1}$ is a basis of $\mathbf{M}_{E \backslash A_1}$. Therefore $M_1 \backslash A_1$

must consist of coloops of $\mathbf{M}'_{E\setminus A_1}$ and $\mathbf{M}' = \mathbf{M}(M_1, A_2, ..., A_r)$. We can repeat for any i, $1 \leqslant i \leqslant r$. Thus

$$\begin{aligned} \mathbf{M}' &= \mathbf{M}(M_1, A_2, ..., A_r) \\ \mathbf{M}' &= \mathbf{M}(A_1, M_2, ..., A_r) \\ &\cdots \\ \mathbf{M}' &= \mathbf{M}(A_1, A_2, ..., M_r). \end{aligned}$$

By the previous lemma, $\mathbf{M}' = \mathbf{M}(M_1, M_2, ..., M_r) = \mathbf{M}$. This completes the proof of the theorem.

COROLLARY 4.17. *Let* $\mathbf{M}$ *be a transversal matroid with maximal presentation* $(M_1, M_2, ..., M_r)$. *Let* C_1 *be a cocircuit of maximum cardinality in* M_1. *Then* $\mathbf{M} = \mathbf{M}(C_1, M_2, ..., M_r)$ *if and only if* $(M_2\setminus C_1, ..., M_r\setminus C_1)$ *has a transversal.*

Proof. By the theorem $\mathbf{M} = \mathbf{M}(C_1, M_2, ..., M_r)$ if and only if (i) $M_1\setminus C_1$ consists of coloops of $\mathbf{M}_{E\setminus C_1}$ and (ii) the rank of $\mathbf{M}_{E\setminus C_1} = \mathbf{M}(M_1\setminus C_1, M_2\setminus C_1, ..., M_r\setminus C_1)$ equals the rank of $\mathbf{M}(M_2\setminus C_1, ..., M_r\setminus C_1)$. Since C_1 is a cocircuit of maximum cardinality in M_1, condition (i) is automatically satisfied, by Lemma 4.8. Thus $\mathbf{M} = \mathbf{M}(C_1, M_2, ..., M_r)$ if and only if (ii) is satisfied. But since C_1 is a cocircuit, the rank of $\mathbf{M}_{E\setminus C_1}$ is $r - 1$. Thus condition (ii) is equivalent to $(M_2\setminus C_1, ..., M_r\setminus C_1)$ having a transversal.

COROLLARY 4.18. *Let* $\mathbf{M}$ *be a transversal matroid with maximal presentation* $(M_1, M_2, ..., M_r)$. *Let* C *be a cocircuit of* $\mathbf{M}$. *Then* C *is a member of some presentation of* $\mathbf{M}$ *if and only if for some* i, $1 \leqslant i \leqslant r$, C *is a maximum cardinality cocircuit of* M_i *with*

$$(M_1\setminus C, ..., M_{i-1}\setminus C, M_{i+1}\setminus C, ..., M_r\setminus C)$$

having a transversal.

Proof. If the conditions are satisfied, then, by Corollary 4.17, C is a member of some presentation of $\mathbf{M}$. Conversely if C is a member of some presentation of $\mathbf{M}$, then after relabeling $\mathbf{M} = \mathbf{M}(C, M_2, ..., M_r)$. Since $\mathbf{M} = \mathbf{M}(M_1, M_2, ..., M_r)$ and $M_1\setminus C$ consists of coloops of $\mathbf{M}_{E\setminus C}$, then, by Corollary 4.9, C is a cocircuit of maximum cardinality in M_1. The result now follows from Corollary 4.17.

The following corollary, due to Bondy, characterizes all minimal presentations.

COROLLARY 4.19 (Bondy [2]). *Let* $\mathbf{M}$ *be a transversal matroid with maximal presentation* $(M_1, M_2, \ldots, M_r)$. *Let* $(C_1, C_2, \ldots, C_r)$ *be a family of cocircuits of* $\mathbf{M}$. *Then* $\mathbf{M} = \mathbf{M}(C_1, C_2, \ldots, C_r)$ *if and only if there exists a permutation* $(i_1, i_2, \ldots, i_r)$ of $\{1, 2, \ldots, r\}$ *such that* C_k *is a maximum cardinality cocircuit in* $M_{i_k}(1 \leqslant k \leqslant r)$ *and*

$$(C_1 \backslash C_k, \ldots, C_{k-1} \backslash C_k, C_{k+1} \backslash C_k, \ldots, C_r \backslash C_k)$$

has a transversal for all k, $1 \leqslant k \leqslant r$.

This is an immediate consequence of Theorem 4.16 and the fact that C is a maximum cardinality cocircuit in M_i if and only if $M_i \backslash C$ consists of coloops of $\mathbf{M}_{E \backslash C}$ (Corollary 4.9).

5. CONCLUDING REMARKS

The preceding section contains much information concerning transversal matroids and their presentations. On the basis of this information there are a number of natural questions which we considered, most of which turned out to have negative answers.

Query 5.1. Can the family $\mathscr{F} = (F_1, \ldots, F_n)$ produced by the algorithm for a rank r matroid have more than r members?

The answer to this question is "yes," as is seen by taking the cycle matroid $\mathbf{M}$ of the complete graph on 4 nodes, K_4. A set of edges of K_4 is in $\mathbf{M}$ if and only if it does not contain a cycle (polygon) of K_4. This matroid has 4 coloop free hyperplanes and rank 3. However all of these hyperplanes are members of $\mathscr{F}$.

Query 5.2. Let $\mathbf{M}$ be a matroid of rank r. Suppose the family $\mathscr{F}$ produced by the algorithm has r members $\mathscr{F} = (F_1, F_2, \ldots, F_r)$. Must $\mathbf{M}$ be a transversal matroid?

The answer to this question is "no," as is seen from the following example. Consider the rank 3 matroid on $E = \{1, 2, \ldots, 7\}$ whose bases are all 3-element sets except for $H_1 = \{1, 2, 3\}$, $H_2 = \{1, 4, 5\}$, $H_3 = \{1, 6, 7\}$. The family produced by the algorithm is (H_1, H_2, H_3). But the inequalities of (4.5) are obviously not satisfied since $1 \in H_1 \cap H_2 \cap H_3$.

Query 5.3. Let M be a transversal matroid with maximal presentation $(M_1, M_2, \ldots, M_r)$. Let C be a maximum cardinality cocircuit in M_1. Must C be a member of some presentation of $\mathbf{M}$?

The answer here again is "no". For an example, let sets $M_1, \ldots, M_5$ be defined by : $M_1 = \{1, 2, 3, 4, 9, 10\}$, $M_2 = \{3, 4, 9\}$, $M_3 = \{1, 2, 9\}$, $M_4 = M_5 = \{2, 4, 5, 6, 7, 8\}$. Let $\mathbf{M}$ be the transversal matroid $\mathbf{M}(M_1, ..., M_5)$. It is easy to verify that $(M_1, ..., M_5)$ is the maximal presentation of $\mathbf{M}$. The set $\{1, 2, 3, 4\}$ is a maximum cardinality cocircuit in M_1 which cannot be a member of a presentation of $\mathbf{M}$ (for, if it were, the element 10 would not be in any set of the presentation).

Under some circumstances the answer to the above query is "yes." If $E\backslash M_1$ is a hyperplane, that is, has rank $r - 1$, then $C = M_1$. If $E\backslash M_1$ has rank $r - 2$, the answer is affirmative and we now prove this. The example given above shows it is not so for rank $r - 3$.

THEOREM 5.4. *Let* $\mathbf{M}$ *be a transversal matroid of rank* r *with maximal presentation* $(M_1, M_2, ..., M_r)$. *Suppose* $E\backslash M_1$ *has rank* $r - 2$. *If* C *is a cocircuit contained in* M_1, *then* C *is a member of some presentation of* $\mathbf{M}$. *If indeed* C *is a maximum cardinality cocircuit in* M_1, *then*

$$\mathbf{M} = \mathbf{M}(C, M_2, ..., M_r).$$

Proof. Suppose there is no coloop-free hyperplane containing $E\backslash M_1$. Then, by the algorithm for some $i \neq 1$, $M_i = M_1$, say $M_1 = M_2$, and C is a maximum cardinality cocircuit in M_1 with $M_1\backslash C = \{x\}$ (x is the coloop of $\mathbf{M}_{E\backslash C}$). But then $(M_2\backslash C, M_3\backslash C, ..., M_r\backslash C) = (\{x\}, M_3\backslash C, ..., M_r\backslash C)$ has a transversal, since $\mathbf{M}_{E\backslash M_1} = \mathbf{M}(M_3\backslash M_1, ..., M_r\backslash M_1)$ has rank $r - 2$. The conclusion then follows by Corollary 4.17. If, on the other hand, there is a coloop-free hyperplane H containing $E\backslash M_1$ (by the algorithm there can be only one), then, by Theorem 4.1, $E\backslash H = M_j$ for some $j \neq 1$, say $E\backslash H = M_2 \subseteq M_1$. If $C = E\backslash H$, then we are done (note that in this case C is not a maximum cardinality cocircuit in M_1). If $C \neq E\backslash H$, then $E\backslash C$ is a hyperplane containing the coloop-free $(r - 2)$-flat $E\backslash M_1$. Therefore $\{E\backslash C\}\backslash\{E\backslash M_1\} = \{x\}$ where x is a coloop of $E\backslash C$ or $M_1\backslash C = \{x\}$. But then $M_2\backslash C = M_2\backslash\{M_1\backslash x\} = \{x\}$ since $M_2 \subseteq M_1$ and $x \in M_2$ (if $x \in E\backslash M_2 = H$, then $H = E\backslash C$). Now $(M_2\backslash C, M_3\backslash C, ..., M_r\backslash C) = (\{x\}, M_3\backslash C, ..., M_r\backslash C)$ has a transversal as before, and the result follows by Corollary 4.17.

Finally we mention the following query:

Query 5.5. Are there at most r different cardinalities for the cocircuits of a transversal matroid $\mathbf{M}$ of rank r? Is every cocircuit contained in some set of the maximal presentation?

The answer to both questions posed is "no." Let $E = \{1, 2, ..., 9\}$ and $M_1 = \{1, 2, 8\}$, $M_2 = \{3, 8, 9\}$, $M_3 = \{3, 4, 5, 6, 7\}$ and let

$$M = (M_1, M_2, M_3).$$

It can be verified that (M_1, M_2, M_3) is the maximal presentation of **M** and that M_1, M_2, M_3 are cocircuits. Hence the cocircuit $C_1 = \{1, 2, 3, 9\}$ is in no $M_i (1 \leqslant i \leqslant 3)$. The set $C_2 = \{4, 5, 6, 7, 8, 9\}$ is also a cocircuit and hence we have cocircuits of cardinalities 3, 4, 5, 6.

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